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PG Sem I

Paper - CC-2

Unit - 2

Topic :- Cauchy Residue theorem

Cauchy Residue theorem

Theorem:- If a single valued fun^n $f(z)$ is analytic within and on a closed contour C, except for a finite number of singular points $z_1, z_2, \dots z_n$ interior to C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\text{Res}}\, f(z) \quad \text{——— (1)}$$

$$= 2\pi i \times (\text{sum of Residues at all Singularities within } C)$$

where the integral is taken in the counter-clockwise sense around C.

Proof:- If a fun^n has only a finite number of singular points in a given domain,

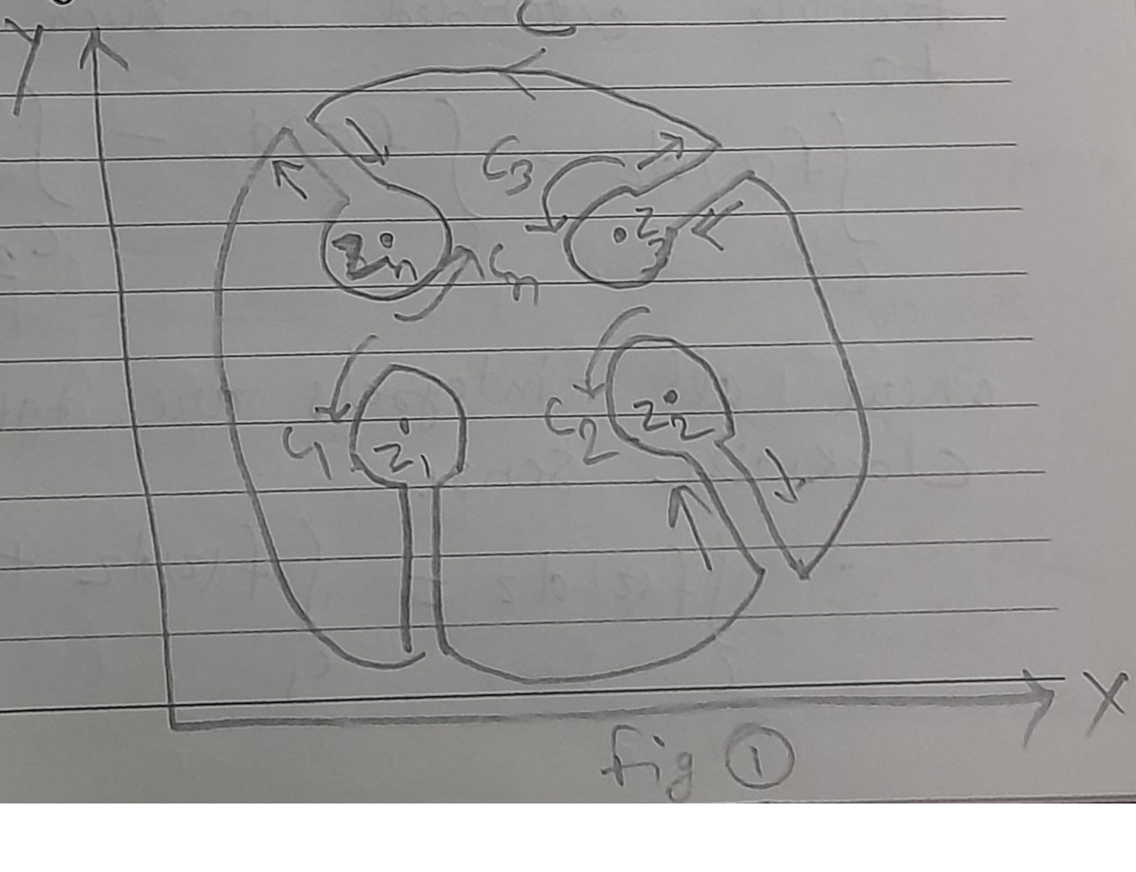

fig (1)

then those singular points are necessarily isolated.

Let a single valued funⁿ $f(z)$ be analytic within and on a closed contour C except for a finite number of singular points $z_1, z_2, \dots, z_n$ interior to C. Let us enclose each of the singular points z_k in a small circle C_k such that these n circles and the contour C are all separated fig ①.

These circles together with the curve C ~~fo~~ form the boundary of a multiply connected domain D within $f(z)$ is analytic. Now deforming the contour C by cross-cuts as shown in fig ①, the Cauchy's integral formula extended to such regions leads to

$$\int_C f(z)\,dz - \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz - \dots - \int_{C_n} f(z)\,dz = 0$$

where all integrals are taken in counter-clockwise sense.

$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz +$$

$$- \cdots + \int_{C_n} f(z)\,dz \quad \text{--- (2)}$$

But $\displaystyle \operatorname*{Res}_{z=z_k} f(z) = \frac{1}{2\pi i}\int_{C_k} f(z)\,dz$

i.e., $$\int_{C_k} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_k} f(z) \quad \text{--- (3)}$$

Therefore equ^n (2) gives

$$\int_{C} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \quad \text{--- (4)}$$

$= 2\pi i \times$ (sum of residues at all singular points with C)

This proves Cauchy residue theorem

It is of extreme importance in evaluating complex and real integrals.

Cor. If a fun^n is analytic in the whole domain except at a finite number of singular points including that at infinity, then the sum of residues at these singularities (including that at infinity) is zero. This can be proved as follows:—

Let C denote the boundary of the domain enclosing all the singularity of analytic funⁿ $f(z)$ except at infinity, then according to Cauchy Residue theorem

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

where $\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$ denotes the sum of all residues at all the finite singularities enclosed by C.

$$\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) = \frac{1}{2\pi i}\int_C f(z)\,dz \quad \text{—(5)}$$

But by definition the residue at infinity

$$\operatorname*{Res}_{z=\infty} f(z) = \frac{1}{2\pi i}\int_{-C} f(z)\,dz = \frac{-1}{2\pi i}\int_C f(z)\,dz \quad \text{—(6)}$$

Adding (5) and (6); we obtain the sum of all the residues of f(z)

$$\sum_{k=1}^{n} \underset{z=z_k}{\text{Res}} f(z) + \underset{z=\infty}{\text{Res}} f(z) = 0 \quad \text{--- (7)}$$